EIDMA

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Lecture 11

- Properties of $\binom{n}{k}$ - cont'd
- Inclusion-exclusion principle
- Pigeon-hole principle

Pasacal's triangle.

Another application of the $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ formula is the Pascal's triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 \searrow & & \swarrow 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & \iddots & \vdots & & & & & \ddots & \\
 & & \vdots & & & & & &
\end{array}
$$

$$
\begin{array}{cccccccccc}
 & \binom{n-1}{0} & \cdots & \cdots & \binom{n-1}{k-1} \searrow & & \swarrow \binom{n-1}{k} & \cdots & \binom{n-1}{n-2} & & \binom{n-1}{n-1} & \\
\binom{n}{0} & & \binom{n}{1} & \cdots & & \binom{n}{k} & & \cdots & & \binom{n}{n-1} & & \binom{n}{n}
\end{array}
$$

Fact.

What happens if we put $a = b = 1$? Nothing unexpected:

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = 2^n = \left|2^{[n]}\right|$$

In other words, the number of subsets in an *n*-element set is equal to the number of 0-element subsets, plus the number of 1-element subsets etc. Surprise, surprise!

Fact.

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$$

Proof.

Just put $a = -1$ and $b = 1$ in the binomial theorem.

Fact.

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{n-k}$$

Proof.

Trivial because for each k, $\binom{n}{k} = \binom{n}{n-k}$

Inclusion-exclusion principle

The rule of addition allows us to express the size of the union of a finite number of finite sets in terms of sizes of individual sets in the case where sets are mutually (pairwise) disjoint. What if they are not?

That's where the inclusion-exclusion principle comes into play.

Theorem (Inclusion-exclusion principle)
For any two sets A *and* B, $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof.

Obviously, $A \cup B = (A \backslash B) \cup B$ and $(A \backslash B) \cap B = \emptyset$.
On the other hand, $A = (A \backslash \mathrm{B}) \cup (A \cap B)$, hence $|A \backslash \mathrm{B}| = |A| - |A \cap B|$. From this we get
$|A \cup B| = |(A \backslash \mathrm{B}) \cup B| = |(A \backslash \mathrm{B})| + |B| = |A| - |A \cap B| + |B|$. QED

The term "inclusion-exclusion" comes from another way of proving this theorem. Namely, writing $|A| + |B|$ we include all elements of $A \cup B$, but those from $A \cap B$ are included (counted) twice, so they must be "excluded" by subtracting $|A \cap B|$ from $|A| + |B|$.

Example. In a group of 25 students 13 chose Algebra as an obligatory course, and 17 chose Ballroom Dancing. How many decided to take both?

Let A and B denote, respectively, the sets of students who chose Algebra and Ballroom Dancing. As we know, $|A| = 13$, $|B| = 17$ and $|A \cup B| = 25$. The inclusion and exclusion principle yields $|A \cup B| = |A| + |B| - |A \cap B|$, hence

$$|A \cap B| = |A| + |B| - |A \cup B| = 13 + 17 - 25 = 5.$$

Example. How many permutations of [6] begin with 1 AND end with 6?

This is really trivial. We place 1 in the first position, 6 in the last position and the remaining 4 digits can be arranged in $4! = 24$ ways.

Example. How many permutations of [6] begin with 1 OR end with 6?

This is a bit harder.
There are 5! permutations beginning with 1 and 5! permutations ending on a 6. There are exactly 4! permutations beginning with 1 AND ending on a 6.
Hence, by the inclusion-exclusion principle, our answer is

$$5! + 5! - 4! = 240 - 24 = 216.$$

What will happen if we apply this strategy to 3 sets, A, B and C? We can consider $A \cup B \cup C$ the union of two sets, $A \cup B$ and C (in fact this is what it is, we right $A \cup B \cup C$ only because $\cup$ is associative) and apply our previous formula recursively: $|A \cup B \cup C| =$
$|(A \cup B) \cup C| =$
$|A \cup B| + |C| - |(A \cup B) \cap C| =$
$|A \cup B| + |C| - |(A \cap C) \cup (B \cap C)| =$
$|A \cup B| + |C| - (|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|) =$
$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

A common-sense explanation is that the elements in A ∩ C ∩ B are included 3 times in $|A| + |B| + |C|$ and excluded 3 times by $(-|A \cap B| - |A \cap C| - |B \cap C|)$ so they must be included again via $+ |A \cap C \cap B|$.

Let $A_1, A_2, \ldots, A_n$ be finite sets. Denote

$S_1 = |A_1| + |A_2| + \ldots + |A_n|$

$S_2 = \sum_{1 \le i_1 < i_2 \le n} \left| A_{i_1} \cap A_{i_2} \right|$ (intersections of pairs)

$\vdots$

$S_k = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \left| A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k} \right|$ (intersections of k-tuples)

Theorem. (generalized inclusion-exclusion principle)

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = S_1 - S_2 + S_3 + \ldots + (-1)^{n+1} S_n$$

Proof.

It can be proved by induction on *n* using the same trick we used to justify the inclusion-exclusion principle for 3 sets. We skip the details. QED

Example.

In how many ways can we hand out 6 (distinguishable) Stingers to 4 guerillas so that each can launch at least one (and we keep none for ourselves)?

Firstly, in how many ways can we give 6 Stingers to 4 guerillas with no additional conditions? Obviously $4^6 = 4096$. From this we must remove those assignments which leave at least one of the guerillas unarmed. Denote by A_i the set of those assignments which give no Stinger to guerilla number *i*. We need to calculate $4096 - |A_1 \cup A_2 \cup A_3 \cup A_4|$, which means we must calculate $|A_1 \cup A_2 \cup A_3 \cup A_4|$. This looks like a job for the inclusion-exclusion principle.

First, for each i $|A_i| = 3^6$ hence $S_1 = 4{\cdot}3^6 = 2916$.
S_2 is a bit more tricky, $|A_i \cap A_j|$ is the number of assignments where i and j don't get a Stinger which means all 6 go to the remaining two people, i.e., $|A_i \cap A_j| = 2^6$. The number of such cases is the number of choices of 2 out of 4 guerillas, $\binom{4}{2}$, so

$S_2 = \binom{4}{2} 2^6 = 6{\cdot}64 = 384$.

Next case is three get nothing, the three can be chosen in $\binom{4}{3}$ ways and all Stingers go to the remaining one – this can only be achieved in one way, so $S_3 = \binom{4}{3} = 4$.

The last case is nobody gets a Stinger – this does not happen; we are supposed to give away all six so $S_4 = 0$.

The final answer is then

$$4^6 - \binom{4}{1} 3^6 + \binom{4}{2} 2^6 - \binom{4}{3} 1^6 + \binom{4}{4} 0^6$$
$$= 4096 - 2916 + 384 - 4 + 0 = 1560$$

How can we translate the last example into mathematish? We assigned Stingers to guerillas, which sounds like we constructed a function from the set of Stingers into the set of guerillas. We also wanted every guerilla to be able to launch a Stinger. This means we wanted our assignments to be onto functions, surjections.

Our solution can be generalized to a formula for number of surjections from an n-element into a k-element set.

Theorem.

The number of surjections form $[n]$ onto $[k]$ is

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n + \ldots + (-1)^{k-1}\binom{k}{k-1}1^n$$

This can be re-shaped into

$$(-1)^0\binom{k}{0}k^n + (-1)^1\binom{k}{1}(k-1)^n + (-1)^2\binom{k}{2}(k-2)^n + \ldots + (-1)^{k-1}\binom{k}{k-1}1^n + (-1)^k\binom{k}{k}0^n =$$

$$\sum_{i=0}^{k}(-1)^i\binom{n}{i}(k-i)^n$$

A trick to remember – the complement principle.
Sometimes it makes sense to enumerate the **complement** of our set and then subtract the result from the size of the universal set.

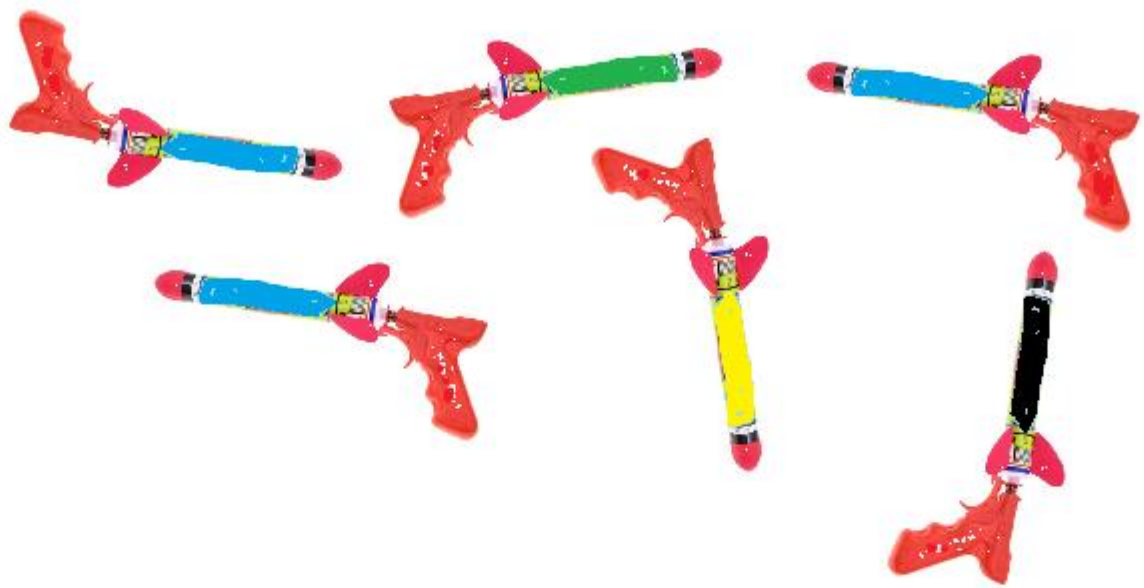
In the last example, in order to count surjections from $[n]$ into $[k]$, we subtracted from the well-known $|[k]^{[n]}|$ the number of non-surjections, which we calculated via inclusion-exclusion principle.

Example. (non-distinguishable case)
In how many ways can we hand-out 6 non-distinguishable Stingers to 4 guerillas so that each will be able to launch at least one (and we keep none for ourselves)?

The task looks similar to that of painting indistinguishable benches, except that we want to use each color at least once.

Guerillas will now play the part of colors (say c_1, c_2, c_3 and c_4) and Stingers play the part of benches. What used to be painting of a bench can now be thought of as painting (labeling) a Stinger with the "color" identifier of a particular guerilla. If x_i denotes the number of Stingers given to c_i we are interested in the number of such solutions of $x_1 + x_2 + x_3 + x_4 = 6$ that for each i, $x_i \geq 1$

In the picture
x_1=3
x_2= $x_3 = x_4 = 1$

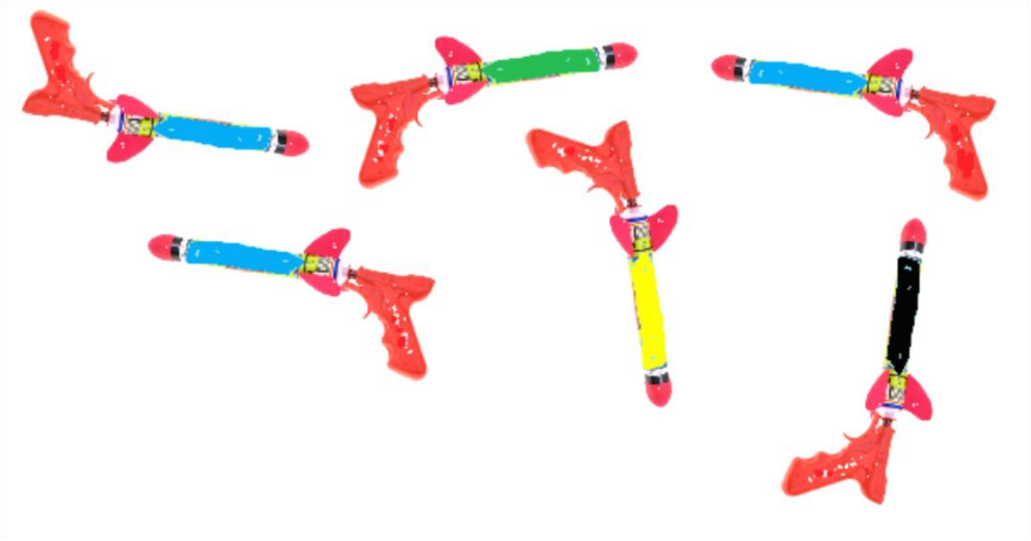


Can we adapt our stars-and-bars strategy to this task?

First, we give a Stinger to each guerilla. Since Stingers are indistinguishable this can be done in one way only and it guarantees that each guerilla will eventually have at least one weapon. Then we distribute the remaining 2 launchers among 4 guerillas using our regular stars-and-bars strategy. This can be done in $\binom{4+2-1}{4-1} = \binom{5}{3} = 10$ ways

(2000,0200,0020,0002,
1100,1010,1001
0110,0101
0011)

The picture illustrates the first solution. Two extra launchers go to blue (c_1) and none to the others.



The last formula can be easily generalized as

Theorem.
The number of k-element subsets with nonzero repetitions of an n-element set is $\binom{k-1}{n-1}$.

Proof. The proof is almost exact copy of the solution of the previous example.

Lesson to remember.
It is more important to understand the method use in the proof than to memorize the theorem. This way you will be able to adjust the theorem to a different purpose and you will be able to reconstruct the theorem in case a memory

Theorem (The pigeon-hole principle)

Let X and Y be two finite sets such that $|Y| = |X|–1$ and $|X| \geq 2$ no function $f: X \rightarrow Y$ is one-to-one.

Proof. (Induction on $n = |X|$)

Part 1.

$n = 1$, Y is empty, there are no functions from X into Y which means no function is one-to-one (and also every function is one-to-one).

$n = 2$ then $|Y| = 1$ and the only function from X into Y maps both element of X to the only element of Y.

Part 2.

We must prove that for every $n \geq 2$,

if

there is no 1-1 function from an n-element X into an n-1-element Y

then

there is no 1-1 function from an $n+1$*-element X into an n-element Y.*

It is complicated precisely because it seems so trivial.

Let $n \geq 2$. Denote $X = \{x_1, x_2, \ldots, x_{n+1}\}$ and let $|Y| = n$.

(By contradiction). Suppose that there exists a 1-1 function $f: X \to Y$. We can label elements of Y with $y_1, y_2, \ldots, y_n$ in such a way that $f(x_{n+1})$ is labeled y_n.

Denote by $X^* = X \setminus \{x_{n+1}\}$ and $Y^* = Y \setminus \{y_n\}$.

Consider $f^* = f|_{X^*}$. We can safely claim that $f^*: X^* \to Y$.

In addition, since $\boldsymbol{f}$ is 1–1 and $\boldsymbol{f(x_{n+1}) = y_n}$, $\boldsymbol{y_n}$ is not the value of $\boldsymbol{f}$ for any element of $\boldsymbol{X}$ other than $\boldsymbol{x_{n+1}}$, we can also claim that $\boldsymbol{f^*: X^* \to Y^*}$ (can you see the difference?). Since every restriction of a 1–1 function is also 1–1, we have constructed a 1–1 function from an n-element X^* into an (n–1)-element Y^* – contrary to the induction hypothesis. QED

In addition, since f is 1–1 and $f(x_{n+1}) = y_n$, y_n is not the value of f for any element of X other than x_{n+1}, we can also claim that $f^*: X^* \to Y^*$.

Why is the emphasis? Because without this part we could not claim that $f^*: X^* \to Y^*$ is a **function** at all so we would not be able to use our *induction hypothesis*.

The pigeon-hole principle may look trivial but it is surprisingly useful in the sense that it allows us to solve problems which look hard or impossible to solve any other way.

Example.

Prove that every n-element set of integers $\{a_1, a_2, \dots, a_n\}$ has a (nonempty) subset whose element sum is divisible by n.

It is trivially true for $n = 1$ and very easy for $n = 2$. However, an attempt to do this by induction is doomed.

Example.

Prove that every n-element set of integers $\{a_1, a_2, \ldots, a_n\}$ has a (nonempty) subset whose element sum is divisible by n.

Solution.

Denote by $f(k) = (a_1 + a_2 + \cdots + a_k)(mod$ n) for k=1,2, … ,n.

Case 1. For some k, $f(k) = 0$. Then our subset is $\{a_1, a_2, \ldots, a_k\}$.

Case 2. For each k, $f(k) \neq 0$. Then f maps $\{1,2, \ldots, n\}$ into $\{1,2, \ldots, n\text{–}1\}$. By the PHP (not the programming language!) there exist p and q, $p < q$, such that $f(p) = f(q)$. This implies that

$(a_1 + a_2 + \cdots + a_q)$– $(a_1 + a_2 + \cdots + a_p) = (a_{p+1} + a_{p+2} + \cdots + a_q)$ is divisible by n, hence our set is $\{a_{p+1}, a_{p+2}, \ldots, a_q\}$. There may exist other such subsets, of course.